Discrete Optimization Studies in Western Europe Map Coloring Based On Findings of Moslem Cartographer In The Early Middle Ages

Noor Saif Muhammad Mussafi[1]

Abstract

In the middle ages, Muhammad Al-Idrisi as a Moslem Cartographer was drawn an incredible map that written in Arabic called Tabula Rogeriana for the Norman King Roger II of Sicily. The study of graph coloring in particular the Welsh-Powell algorithm is expected to provide an alternative perspective to such map. This study examines the discrete optimization of map coloring based on geographic using chromatic number upper bound approach. Data on the original map that drawn on single color, will be represented in the graph by considering the geographical aspect. Furthermore it needs graph coloring analysis by means of Welsh-Powell algorithm. The results provide an alternative model of optimal Western Europe multicolor map.

Keywords: Discrete Optimization, Muhammad Al-Idrisi, Tabula Rogeriana, Welsh-Powell Algorithm

1. Introduction

Since the beginning of the century until the mid-century there has been no discovery of Ptolemy's map except map as good as have been made a geographer who were born in Ceuta Spain, namely Sharif Al-Idrisi (1154). By the support of Norman King Roger II, Al-Idrisi, known as the geography teacher for European people needs over 14 years to collect all information e.g. area, climate, rivers, lakes, mountains, beaches, soil, and socio-cultural conditions that produce innovation product called Tabula Rogeriana.[2] Indeed, what is explored by Al-Idris in agreement with the explanation in the Qur'an, verse 97 of Surah An-Nisa.

قَالُوٓاْ أَلَمۡ تَكُنۡ أَرۡضُ ٱللَّهِ وَٰسِعَةٗ فَتُهَاجِرُواْ فِيهَاۚ

This verse states that World is spacious, not only Mecca. Therefore, in order to obtain optimal result, he made serious effort to collect the above data from anyone whom he met not only in Sicily Europe.

---

[1] *Mathematics Department, Faculty of Science and Technology, UIN Sunan Kalijaga. Mail Address: Jl. Marsda Adisucipto 1 Yogyakarta 55281. Email: om_norsa@yahoo.com. Presented at Poster Session of AICIS-14 Balikpapan November 21th – 24th, 2014.*

[2] Idrisi, Ash-Sharif (2008). *In Encyclopædia Britannica*. Retrieved April 6, 2008, from Encyclopedia Britannica Online.

Although Tabula Rogeriana is only a world map with single color but it is considered accurate by the cartographer. In cartography (the study and practice of making maps on the premise that reality can be modeled in ways that communicate spatial information effectively), to mark the area, the river, the mountains, and the sea can be used with no limit number of colors i.e. multicolor. And now the problem becomes how many colors are actually needed to color the map? In the concept of graph theory, a planar graph is identical to the representation of the map. Furthermore, in order to minimize the number of colors used in graph (map) it can be used Welsh-Powell algorithm and graph coloring theorem.

Welsh-Powell algorithm is one of the techniques in the greedy algorithm to find the chromatic number of discrete optimization approach that typically prioritizes coloring vertices in the order of highest degree.[3] While the graph coloring theorem to currently experiencing some growth. The first theorem states that a planar graph has chromatic number ≤ 6. The second theorem[4] states that a planar graph has chromatic number ≤ 5. Until last theorem finally found in about the 19th century states that a planar graph has chromatic number ≤ 4. Theorem invented by Apple and Haken has been successfully analyzed almost 2,000 graphs involving millions of cases. In this study, graph coloring of Tabula Rogeriana restricted to the region of Western Europe. The analysis used is an upper bound for the chromatic number and the four-color theorem.[5]

2. Graph

A graph is a pair $G = (V, E)$ of sets such that $E(G) \subseteq \binom{V(G)}{2}$; thus, the elements of E are 2-element subsets of V or unordered pair of V. The elements of V are the vertices (or nodes) of the graph G, the elements of E are its edges (or lines). The usual way to picture a graph is by drawing a dot for each vertex and

---

[3] Gross, Jonathan L. and Yellen Jay (2003). *Handbook of Graph Theory*. CRC Press.

[4] Wilson, Robert A. (2002). *Graph Coloring and The Four Colour Theorem*. New York: Oxford University Press.

[5] K. Appel and W. Haken (1977). *The Solution of The Four-Color-Map Problem*. Sci. Amer. 237, 108-121.

joining two of these dots by a line if the corresponding two vertices form an edge.[6] In a graph, two or more edges joining the same pair of vertices are multiple edges, while an edge joining a vertex to itself is a loop. A graph with no multiple edges or loops is a simple graph.[7] Figure 1 shows a non-simple graph $G = (V, E)$ with V(G) = {a, b, c, d, e, f} and E(G) = {(a,a), (a,b), (a,c), (a,e), (c,d), (c,f)}.

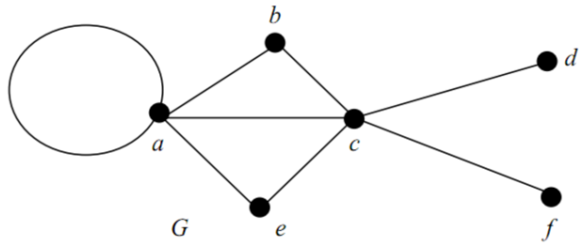

Figure 1. Graph *G*

A vertex *v* is incident with an edge *e* if $v \in e$; then *e* is an edge at *v*. Two vertices *x,y* of *G* are adjacent, or neighbours, if *xy* is an edge of *G*. If all the vertices of *G* are pairwise adjacent, then *G* is complete. A complete graph on *n* vertices is a K_n; a K_3 is called a triangle. The degree (or valency) $d(v)$ of a vertex is the number of edges at *v*; this equal to the number of neighbours of *v*. A loop has a vertex of degree 2 and a vertex of degree 0 is isolated. In figure 1, d(a)=5, d(b)=2, d(c)=5, d(d)=1, d(e)=2, and d(f)=1, so that the number of vertex degree of *G* is 16. A graph *G* is planar if it can be drawn in the plane such way that no two edges meet except at a vertex with which they are both incident. Every plane drawing of a planar graph divides the plane into a number of regions, called faces.[8] For example, a planar graph K_4 or tetrahedron has four faces (figure 2) since it divides the plane into four regions – three traingles (3-cycles) and one "infinite region". A planar graph *G* has graph *G** which geometrically called dual of planar graph *G*. In figure 3 above illustrate a planar graph (which identically can be represented as map M) and its dual graph.

---

[6] Diestel Reinhard (2005). *Graph Theory*. Third Edition. Berlin: Springer-Verlag.

[7] Joan Aldous & Robin Wilson (2004). Graph and Application: An Introductory Approach. 4th edition. Springer.

[8] Ibid 7, p.248

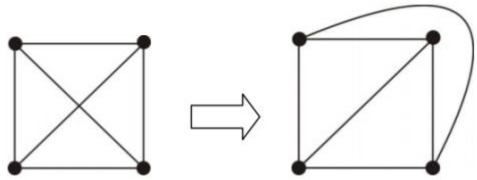

Figure 2. Planar Graph K_4

There are two step to convert a map M into dual graph[9]: (1) consider a vertex in each region as representative i.e. capital region and (2) if two arbitrary region are in borders then connect the verices in question with an edge.

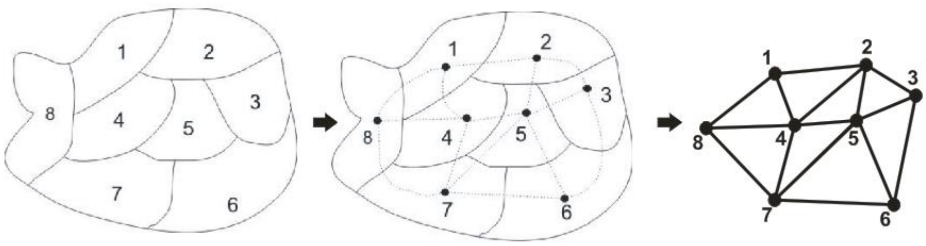

Figure 3. Planar Graph (map M) and its dual graph

3. Graph Coloring

a. Vertex Coloring

How many colors needed to color the countries of a map in such a way that adjacent countries are colored differently? A vertex coloring of a graph $G = (V, E)$ is a map $c: V \rightarrow S$ such that $c(v) \neq c(w)$ whenever *v* and *w* are adjacent, i.e. no two adjacent vertices have the same color.[10]

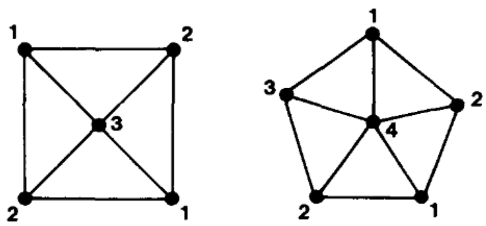

Figure 4. Graph G_1 and G_2

The elements of the set S are called the available colors. If S has smallest cardinality k such that G has a k-coloring, then $c: V \rightarrow \{1,2, \dots, k\}$ is a vertex coloring.[11]

---

[9] Slamet, S. and Makaliwe, H. (1991). *Matematika Kombinatorik*. Jakarta: Gramedia.

[10] Hartsfield and Ringel (1994). *Pearls in Graph Theory: A Comprehensive Introduction*. 2nd edition. San Diego. CA Academic Press.

[11] Jensen, T.R. and Toft, B. (1995). *Graph Coloring Problems*. Wiley-Interscience. New York.

This k is the chromatic number of G, denoted by $\chi(G)$. Clearly $\chi(G) \leq |V(G)|$, an easy way to find the lower bound is by finding complete sub graph in G and the upper bound of chromatic number mostly known. In figure 4, the chromatic number of G_1 is three and respectively G_2 is four.

Theorem 3.1

If G is simple graph with maximum vertex of degree $\Delta(G)$, then $\chi(G) \leq \Delta(G) + 1$

Thus bound is developed by Brook as follow.

Theorem 3.2

Let G be connected graph. If G is neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

b. Four Color Theorem

In order to proof four color theorem, it need to know the following Euler Polyhedral Formula and its properties.

Theorem 3.3 (Euler's Formula)

Let G be a connected plane graph with n vertices, m edges, and f faces. Then

$$n - m + f = 2$$

Corollary 3.4

Let G be a simple connected plane graph with n (≥ 3) vertices and m edges. Then $m \leq 3n - 6$

Proof:

Consider a plane drawing of a simple connected planar graph G with f faces. Since a simple graph has no loops or multiple edges, the degree of each face is at least 3. It follows from the handshaking lemma for planar graphs that

$$3f \leq 2m$$

Substituting for f from Euler's formula $f = m - n + 2$, we obtain

$$3m - 3n + 6 \leq 2m$$

And hence $m \leq 3n - 6$ as required.

Theorem 3.5

The chromatic number of a planar graph is at most four

Proof:

This short proof was represented by Peter Doerre.[12] Without loss of generality graphs to be studied for the proof can be restricted to triangulations. A planar graph G is called a triangulation if it is connected, without loops, and every interior face is (bounded by) a triangle (3-cycle), as well as the (in the plane infinite) exterior face. It follows from Euler's formula that a planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges, and the triangulations are the edge maximal planar graphs. Every planar graph H (e.g. a proper near triangulation) can be generated from a triangulation G by removing edges and disconnected vertices. As removal of edges reduces the number of restrictions for coloring, the chromatic number of H is not greater than that of G.

c. *Greedy Algorithm*

There are several techniques in *k-coloring* of graph G based on heuristic, one of which is greedy algorithm that had been criticized its probability by Kucera.[13]

Theorem 3.6

Let G be a simple graph whose maximum vertex degree is d. Then

$$\chi(G) \leq d + 1$$

The above theorem means that the upper bound of chromatic number of those graph G is $d + 1$.

In the context of graph coloring, for some ordering of the vertex degree, color the nodes in this order with the smallest color that is possible without violating constraints.[14]

The algorithm which actually used in this research is Welsh-Powell that is part of greedy coloring. Purpose of using algorithm is to find chromatic number of graph G. This algorithm classified in type of Largest Degree Ordering (LDO) algorithm which prioritizes coloring vertices in the order of highest degree. According to Welsh and Powell[15], the steps are: (1) sort the vertices of a graph G in a decreased degree of which the largest (order as

[12] Slamet, S. and Makaliwe, H. (1991). *Matematika Kombinatorik*. Jakarta: Gramedia.

[13] L. Kucera (1991). *The Greedy Coloring is a Bad Probabilistic Algorithm*. Journal of Algorithms, 12:674-684.

[14] A.E. Eiben (1998). *Graph Coloring with Adaptive genetic Algorithm*. CiteSeer^x. Digital Library.

[15] D.J.A. Welsh and M.B. Powell (1967). *An Upper Bound for the Chromatic Number of a Graph and Its Application to Timetabling Problems*. The Computer Journal. vol.10. pp. 85–86.

the vertices of a graph G in a decreased degree of which the largest (order as is possible is not unique because several nodes to the same degree), (2) use one color for the coloring the first vertex (the vertex with the highest degree) and the other vertices (in sequential order) that are not adjacent to the first vertex, (3) start again with the next highest vertex degree in the ordered degree that has not been colored and repeat the vertex coloring process using a second color, (4) repeat the previous three steps until all vertices have been stained.

4. Findings of Muhammad Al-Idrisi (Tabula Rogeriana)

The Book of Roger or popularly referred to *Tabula Rogeriana* (figure 5) is a description of the world i.e. world map created by the Arab cartographer, Muhammad Al-Idrisi, in 1154. Al-Idrisi worked on the commentaries and illustrations of the map for some years at the court of the Norman King Roger II of Sicily, who commissioned the work around 1138.[16] The book, written in Arabic, is divided into seven climate zones (in keeping with the established Ptolemaic system), each of which is sub-divided into ten sections, and contains maps showing the Eurasian continent in its entirety, but only the northern part of the African continent. The map is oriented with the North at the bottom, but officially has been copied into North oriented up.

Harley and Woodward[17] added that to produce the work, al-Idrisi interviewed experienced travelers individually and in groups on their knowledge of the world. Roger II had his map engraved on a silver disc weighing about 300 pounds. It showed, in al-Idrisi's words, "the seven climatic regions, with their respective countries and districts, coasts and lands, gulfs and seas, watercourses and river mouths".

In 1904, on the work of Al-Idrisi, S. P. Scott also commented that the compilation of Edrisi marks an era in the history of science. Not only is its historical information most interesting and valuable, but its descriptions of many

[16] Hubert Houben (2002). *Roger II of Sicily: A Ruler Between East And West*. Cambridge University Press. UK.

[17] J. B. Harley and David Woodward (1992). *The History of Cartography Volume 2, Book 1: Cartography in the Traditional Islamic and South Asian Societies*. Chicago and London: University of Chicago Press. ISBN 0-226-31635-1.

Figure 5. Tabula Rogeriana (North Pole)

parts of the earth are still authoritative. For three centuries geographers copied his maps without alteration. The relative position of the lakes which form the Nile, as delineated in his work, does not differ greatly from that established by Baker and Stanley more than seven hundred years afterwards, and their number is the same. The mechanical genius of the author was not inferior to his erudition. The celestial and terrestrial planisphere of silver which he constructed for his royal patron was nearly six feet in diameter, and weighed four hundred and fifty pounds; upon the one side the zodiac and the constellations, upon the other-divided for convenience into segments-the bodies of land and water, with the respective situations of the various countries, were engraved.

5. Result

Al-Idrisi invention through the world map, especially Europe and the Middle East known as the Tabula Rogeriana, still represented in one color. Obviously a map will be easier to understand when signified in multicolor. Therefore this study attempted to color the work of Al-Idrisi, particularly Western Europe, using discrete optimization approach such that to obtain the minimum amount of staining. The following are some steps in the coloring Tabula Rogeriana using vertex coloring principles:

1. Label the names of the countries on the Tabula Rogeriana along boundaries between countries. This is done to make the dual graph on the map.
2. Transform Tabula Rogeriana into graph representation. As explained on theoritical framework, graph consists of vertices and edges. Vertices represent

the countries in Western Europe while the edges denote two arbitrary state borders. Then the vertices that are adjacent are connected.

3. Coloring each vertex of the graph by using the Welsh-Powell algorithm to obtain the chromatic number (the minimum number of colors needed to color Tabula Rogeriana).
4. Specifies an alternate presentation of the colored map of Tabula Rogeriana.

From Tabula Rogeriana and European Union (EU), it could be identified 17 countries in Western Europe i.e. Portugal (01), Spain (02), France (03), Italy (04), Switzerland (05), Luxemburg (06), Belgium (07), Netherland (08), Germany (09), Austria (10), Greece (11), Denmark (12), Norway (13), Sweden (14), Finland (15), United Kingdom (16), and Ireland (17).

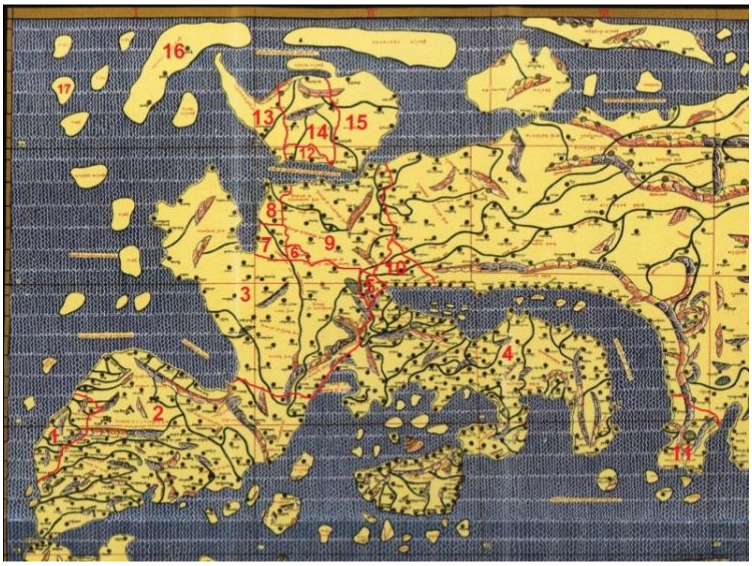

Figure 6. Western Europe with 17 countries

Here the edges indicate two arbitrary country borders e.g. Germany connected by an edge to Switzerland, Luxemburg, Belgium, Netherland, Austria, and France. The vertices (v_i, for $1 \leq i \leq 17$) specify the capital town of each country as the following table:

v_i	**Country**	v_i	**Country**	v_i	**Country**
v_1	Portugal	v_7	Belgium	v_{13}	Norway
v_2	Spain	v_8	Netherland	v_{14}	Sweden
v_3	France	v_9	Germany	v_{15}	Finland
v_4	Italy	v_{10}	Austria	v_{16}	United Kingdom
v_5	Switzerland	v_{11}	Greece	v_{17}	Ireland
v_6	Luxemburg	v_{12}	Denmark		

Table 1. List of Vertices

Based on table 1 and figure 6 that show Western Europe countries except Iceland, convert into dual graph as illustrated in figure 7.

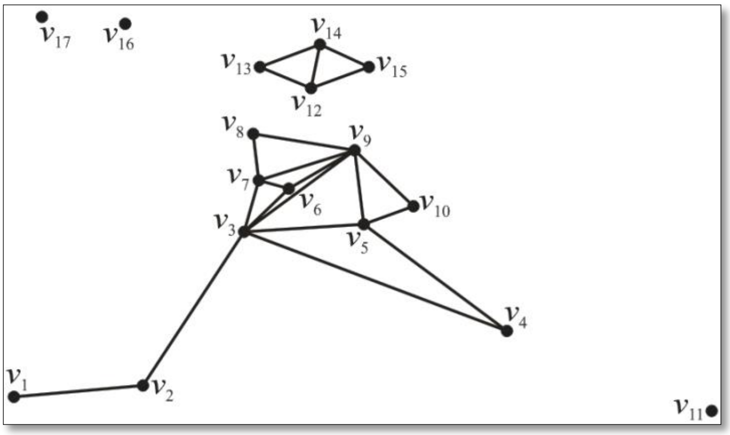

Figure 7. Dual Graph of Figure 5

Formally from the dual graph above, the map can be colored by utilizing principle of the four color theorem and Welsh-Powell algorithm. **First**, from figure 7 it can be analyzed each vertex by descending the degree of vertex, in succession displayed in table 2.

	$v_3 = v_9$	$v_5 = v_7$	$v_{14} = v_{12} = v_6$
d (v_i)	**6**	4	3
	$v_2 = v_4 = v_8 = v_{10} = v_{13} = v_{15}$	v_1	$v_{11} = v_{16} = v_{17}$
d (v_i)	2	1	0

Table 2. List of vertex degree (descending)

Second, consider a color e.g. green to color the first vertex (has six of vertex degree), France (v_3) together with the non-adjacent vertex to France, that is Portugal (v_1), Netherland (v_8), Switzerland (v_{10}), Greece (v_{11}), Norway (v_{13}), Finlad (v_{15}), United Kingdom (v_{16}), and Ireland (v_{17}). **Third**, continue to color the next highest degree of vertex (non-green) e.g. sky blue. The next vertex in question is Germany (v_9) together with the non-adjacent vertex of Germany, that is Spain (v_2), Italy (v_4), and Sweden (v_{14}). **Fourth**, continue coloring the next highest degree of vertex (non-sky blue) e.g. Netherland (v_7) together with non-adjacent vertex of Netherland, that is Austria (v_5) and Denmark (v_{12}). Here we use the red once. **Fifth**, the only vertex which not colored yet is Luxemburg. So we add a new color of yellow at Luxemburg (v_6). The algorithm is stopped since all vertices have been colored (figure 8).

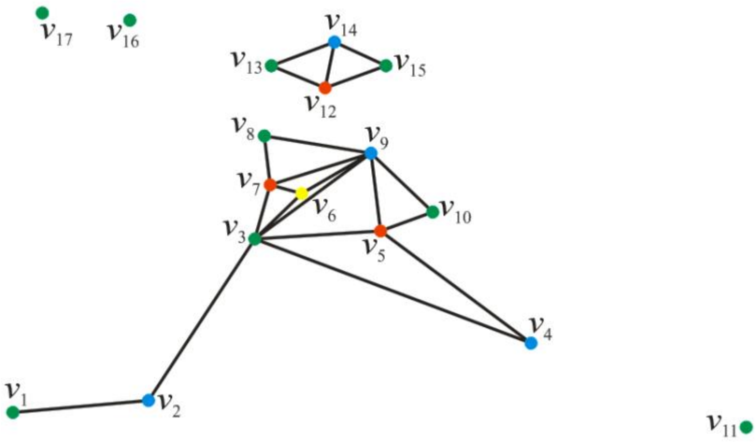

Figure 8. Graph Coloring with Chromatic Number of Four

The last step in this research is to convert graph coloring in figure 8 into map representation as well as Tabula Rogeriana. The results in figure 9 offer an alternative model of optimal Western Europe multicolor map.

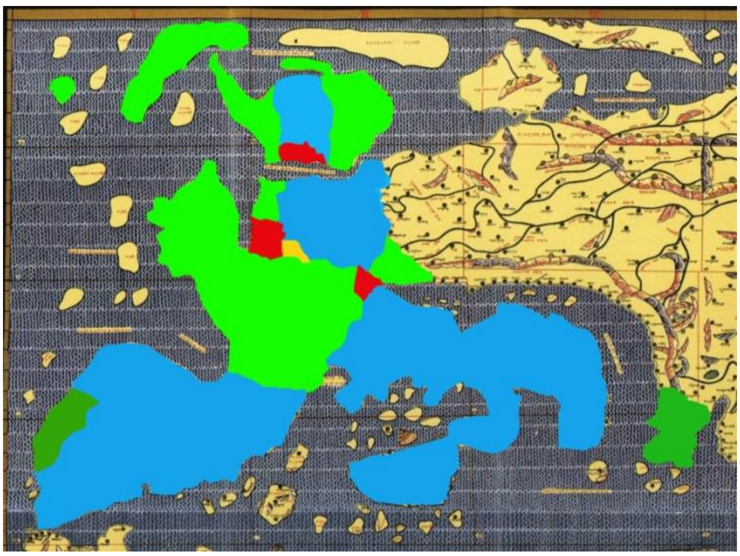

Figure 9. An optimal Western Europe multicolor map based on Tabula Rogeriana

6. Conclusion

Tabula Rogeriana can be analyzed in view of discrete optimization particularly graph coloring which principally seeking for upper bound chromatic number. It can be concluded as well as four color theorem that the chromatic number of a planar graph is at most four. The result provide an alternative Western Europe map coloring of Tabula Rogeriana. This study can be done by modeling the countries in Western Europe that illustrated in Tabula Rogeriana into dual graph, then analyze the dual graph using the principle of Welsh-Powell Algorithm in order to acquire a minimum color.

7. References

A.E. Eiben (1998). *Graph Coloring with Adaptive genetic Algorithm*. CiteSeer^x. Digital Library.

Appel and Haken (1989). *Every Planar Map is Four-Colorable*, Providence, RI: American Mathematical Society, ISBN 0-8218-5103-9.

Bondy, J.A. and Murty, U.S.R. (1976). *Graph Theory With Applications*. New York: MacMillan Press.

Diestel Reinhard (2005). *Graph Theory*. Third Edition. Berlin: Springer-Verlag.

D.J.A. Welsh and M.B. Powell (1967). *An Upper Bound for the Chromatic Number of a Graph and Its Application to Timetabling Problems*. The Computer Journal. vol.10. pp. 85–86.

Gross, Jonathan L. and Yellen Jay (2003). *Handbook of Graph Theory*. CRC Press.

Hamdy A. Taha (2007). *Operations Research: an Introduction*. Prentice Hall. NJ.

Hartsfield and Ringel (1994). *Pearls in Graph Theory: A Comprehensive Introduction*. 2nd edition. San Diego. CA Academic Press.

Heawood, P. J. (1890). *Map-Colour Theorem*. Quarterly Journal of Mathematics, Oxford 24: 332–338.

Hubert Houben (2002). *Roger II of Sicily: A Ruler Between East And West*. Cambridge University Press. UK.

Idrisi, Ash-Sharif (2008). *In Encyclopædia Britannica*. Retrieved April 6, 2008, from Encyclopædia Britannica Online.

J.A, Boundy and U.S.R, Murty (1976). *Graph Theory with Application*. New York : North-Holan.

J. B. Harley and David Woodward (1992). *The History of Cartography Volume 2, Book 1: Cartography in the Traditional Islamic and South Asian Societies*. Chicago and London: University of Chicago Press. ISBN 0-226-31635-1.

Jensen, T.R. and Toft, B. (1995). *Graph Coloring Problems*. Wiley-Interscience. New York.

Joan Aldous & Robin Wilson (2004). *Graph and Application: An Introductory Approach*. 4th edition. Springer.

K. Appel and W. Haken (1977). *The Solution of The Four-Color-Map Problem*. Sci. Amer. 237, 108-121.

L. Kucera (1991). *The Greedy Coloring is a Bad Probabilistic Algorithm*. Journal of Algorithms, 12:674-684.

Mohammed Hamidullah (1968). *Muslim Discovery of America before Columbus*. Journal of the Muslim Students' Association of the United States and Canada 4 (2): 7-9.

Peter Doerre (2013). *An Almost Trivial Proof of The 4-Colour Theorem*. Department of Informatics and Natural Sciences FH Südwestfalen Germany.

Siang, Jong Jek. (2002). *Matematika Diskrit dan Aplikasinya pada Ilmu Komputer*. Yogyakarta: Andi.

Slamet, S. and Makaliwe, H. (1991). *Matematika Kombinatorik*. Jakarta: Gramedia.

Walter Klotz (2002). *Graph Coloring Algorithms*. Mathematik-Bericht 5, 1-9, TU Clausthal. Germany.

Wilson, Robert A. (2002). *Graph Coloring and The Four Colour Theorem*. New York: Oxford University Press.

Winston, Wayne L. (2000). *Operation Research: Applications and Algorithms*. Fourth Edition. Canada: Thomson.

____________ (2012). *Composition of macro geographical (continental) regions, geographical sub-regions, and selected economic and other groupings*. United Nations Statistics Division. Accessed July 27, 2013.